A Theory of Tax Effects on Economic Damages

Scott Gilbert
Southern Illinois University Carbondale

Comments? Please send to gilberts@siu.edu

November 29, 2012

Abstract

This note provides a theoretical statement about the effect of tax on the present value of lost income streams. I consider the simple case of flat tax rates on earnings and interest income. I approximate tax effects via the instantaneous rate of change - in present value – when the tax rate goes from zero to a small positive number. In this setting I show that present value is lower before tax than after tax when the earning stream is short, with the reverse outcome holding when the earnings stream is long. The switch point, where the tax effect goes from negative to positive, depends on the theoretical model's inputs. I characterize the effect of inputs on this switch point, and illustrate via an example of an injured railroad worker's claim of economic damages.

Keywords: income stream; tax; present value; tort; personal injury; wrongful death

1. Introduction

A common understanding in the legal community is that the tax effects on personal injury damage awards are more "plaintiff issue" when the plaintiff is young, and more a "defense issue" when the plaintiff is old. The reasoning is that, for a young person with a long future period of growing incomes, if awards are adjusted for tax then the positive effects of interest tax inclusion come to outweigh the negative effects of income tax inclusion, whereas the reverse is thought to hold for an older person. This understanding is based on actual experience at trials, with damage estimates compared pre-tax and post-tax, and so forms a sort of "empirical rule".

In terms of economic theory, Anderson and Barber (2010) take up the task of proving the validity of the "empirical rule" of tax effects on growing income. Gilbert (2012) reports counterexamples to this theory, but also describes a scenario where the logic might work, namely when earnings are taxed at a small (percentage) rate. The arguments in Gilbert (2012) are informal, and the present work aims to restate them more formally and in greater depth.

In the remainder of this work, Section 2 (Tax Effects) presents the main result about tax effects, and Section 3 describes the switch-point at which tax effects go from negative to positive. Sections 4 and 5 characterize the link between the theoretical model's inputs – rates earnings growth and interest – on the switch-point. Section 6 applies the theory to an actual legal case involving an injured railroad worker, and Section 7 concludes. Proofs of mathematical results appear in the Appendix.

2. Tax Effects

As in Gilbert (2012), consider a future pre-tax earnings stream $E_1, E_2, \ldots$, in future periods 1, 2,…, N growing at a constant rate g. In period 0, the present value of the earnings stream is the lump sum of money which, when invested at the risk-free interest rate r, generates the earnings stream. If neither interest nor earnings are taxed then present value is "before tax", while if both interest and earnings are taxed at the same (constant) rate τ then present value is "after tax".

The issue is whether the introduction of tax lowers or raises the present value of earnings streams. Gilbert (2012) sketches an argument in support of the idea that tax lowers present value for older workers (with shorter anticipated income streams) while raising it for younger workers (with longer anticipated income streams), provided that the tax rate is close to zero. The following theorem provides a formal basis for that claim.

Theorem 1: *Suppose that a person suffers an income loss that deprives them of earnings that would have grown at a constant positive rate g over a finite worklife. Suppose also that the interest rate r is constant, and that tax rates on earnings and interest are equal. Then the effect of tax on present value switches from negative to positive at some horizon N^*: for any $T > N^*$ and all tax rates τ sufficiently small, the present value of the earnings stream is lower after tax than before tax when horizon $N < N^*$, but is higher after tax when $N^* < N < T$.*

For an older worker, the earnings horizon N is relatively short, and Theorem 1 says that the present value of this worker's income stream is negatively impacted by the deduction of tax from earnings and interest. For a younger worker, Theorem 1 says the opposite. In both cases, the result is true only for tax rates τ that are sufficiently small.

3. Switch-Point

The theory in Section 2 establishes the existence of an earnings horizon N^* at which tax effects on present value switch from negative (for shorter horizons) to positive (for longer horizons). To be more explicit about this switch-point N^*, let me briefly summarize the logic underlying Theorem 1.

As detailed in the Appendix, the instantaneous effect of tax on present value present value (PV) is given by the derivative of derivative of PV with respect to τ . Computing this derivative, and setting it equal to zero, establishes a situation where the instantaneous tax effect is zero – as in equation A.2 in the Appendix. This equation can be usefully recast in terms of Macaulay duration[1] D:

(1)
$$D = \frac{\sum_{t=1}^{N} \frac{tE_t}{(1+r)^t}}{\sum_{t=1}^{N} \frac{E_t}{(1+r)^t}}$$

which measures the (weighted) average waiting time until future (pre-tax) earnings are received. In terms of D, N^* is the closest integer approximation to the following equation – which is also equation A.14 in the Appendix:[2]

(2)
$$D = 1 + \frac{1}{r}$$

[1] See Macaulay (1938) and also Hicks (1939).

[2] If there are two such best approximations then they are successive integers. Let N^* be the smaller one.

A closed-form solution N^* to the switch-point equation (2) is not generally available, but consider the case of total offset, where earnings growth rate g equals the interest rate r. Then duration D equals time horizon N, in which case (2) has solution:

(3) $$N^* = 1 + \frac{1}{r}$$

4. **The role of earnings growth**

Having characterized the effect of tax on the present value of growing income streams, it is useful to consider how the rate g of earnings growth influences the result. The value of g is potentially important. If there is no growth ($g = 0$) then tax adjustment lowers present value – see Samuelson (1964), regardless of the earning horizon N. On the other hand, with growth ($g > 0$)the effect of tax is mixed, lowering present value at short horizons, raising it at longer horizons, in the sense described by Theorem 1. One way to rationalize these disparate results is to conjecture that the switch-point N^*, where tax effects go from negative to positive, depends on the growth rate g and increases as g falls toward 0. The remainder of this section is devoted to confirming this conjecture.

To determine the effect on N^* of changes in g, I'll use the switch-point equation (2) in the previous Section. In that equation, N^* is the value of N that equates duration D to a function of the interest rate. To see how g changes N^*, we can first see how g changes D. To this end define the net growth rate of earnings, as in Gilbert (2011, equation 8):

(4) $$h = \frac{1+g}{1+r} - 1$$

Then Macaulay's duration D takes the form:

$$D = \frac{\sum_{t=1}^{N} (1+h)^t t}{\sum_{t=1}^{N} (1+h)^t} \tag{5}$$

It is then possible to determine the effect of changes in h and N on D, as follows:

Lemma 1: Duration D *is increasing in horizon* N *and net growth rate* h.

Returning to the original problem, we can link a change in g to a change in h, a change in h to a change in D, and a change in D to a change in the solution N^* to the switch-point equation (2). I state the result as a theorem, with details in the Appendix.

Theorem 2: *Under the conditions of Theorem 1, the switch-point earnings horizon* N^* *lengthens when earnings grow at a slower rate* g.

For a worker with low projected earnings growth, Theorem 2 says that tax adjustment will have a negative effect on present value unless the earnings horizon is long.[3] With a higher growth rate, positive tax effects become more plausible.

5. **The role of interest**

Having linked earnings growth to the tax effect switch-point N^*, we can similarly attempt to link the interest rate r to N^*. To this end we can try adapt the logic of Section 4, by showing that a change in r

[3] As earlier, tax effects are asserted only for tax rates τ that can be considered sufficiently small that Theorem 1 applies.

changes net growth rate h, a change in h changes duration D, and a change in D changes the solution N^* to the switch-point equation (2). Indeed, an increase in r clearly decreases h, and by Lemma 1 this decreases D, causing a drop in the left-hand side of (2). But it also causes the right-hand side of (2) to fall, in which case the ultimate effect on equation-balancing N^* is not evident.

Taking a different route, note first that since duration D is a (weighed) average of time periods, it is no larger than the time horizon N. This fact, together with the switch-point condition (2), implies a bound on switch-point N^* in terms of the interest rate:

(6) $$N^* \geq 1+\frac{1}{1+r}$$

According to (6), if the interest rate falls toward zero then the switch-point increases to infinity. In other words, if the interest rate is low then tax effects on present value are negative except at long horizons.

The bound (6) on N^* is sharp, as it is achieved as an equality in the case of total offset between earnings growth and interest – see (3). With total offset, there is a negative relationship between r and N^*. The relationship holds more generally, and can be shown by taking the derivative of present value PV with respect to both τ and r, then signing it. I state the result as follows, with proof provided in the Appendix.

Theorem 3: *Under the conditions of Theorem 1, the switch-point earnings horizon N^* is decreasing in the interest rate r.*

6. Example

To illustrate the theory set forth in Sections 2 through 5, consider the following legal case. In the year 2005 a railroad worker was hurt on the job in the state of Illinois. The worker, a 48 year-old male, sued a railroad company for loss of future income caused by his injury on the job[4]. A trial was held in the year 2007. As this is a railroad case it falls under FELA[5] rules, with lost earnings measured after-tax. An economist served as expert witness and estimated economic loss, and assumed an earnings growth rate g = 0.051, an interest rate r = 0.066, earnings horizon N = 17, tax rate τ = 12.23%, and base salary E_0 = \$61,061.

To illustrate tax effects, in Table 1 I show the present value of the railroad worker's lost future income stream on pre-tax and post-tax bases, for earnings horizon N ranging from 1 through 40 years, with base salary normalized[6] to \$1 and with equal tax rates on earnings and interest. For horizons 1 through 34 years, subtraction of tax has a negative effect on present value. For horizon 35 and beyond, the reverse holds.

According to Theorem 1, if the tax rate τ is sufficiently small then tax effects should switch signs at horizon N^* defined via the breakeven condition (2). To determine N^* the last two columns show relevant inputs, including Macaulay-Hicks duration D and the difference $D-\left(1+\frac{1}{r}\right)$ whose sign switches at N^*. According to the table, N^* = 34. Theorem 1 predicts that tax will have negative effects

[4] This case, in which I had no involvement, was tried in the First Judicial Circuit of the Illinois state court, details available upon request.

[5] Federal Employers Liability Act.

[6] That is, each normalized present value in Table 1 must be multiplied by actual base salary (\$61,061) to get actual present value.

at horizons 1 through 34, positive effects at horizons 35 up to some point T whose value remains unspecified. The theory is consistent with the actual present value results in Table 1.

According to Theorem 2, a lowering of the earnings growth rate g should raise the switch-point time horizon N^*. As a check, when I cut the value $g = 0.0512$ in half, N^* rises to 42. When I cut g in half again, N^* rises to 52, consistent with Theorem 2. Similarly, when the initial g is doubled, N^* falls to 27, and when g is quadrupled N^* falls to 22.

According to Theorem 3, the switch-point N^* should fall when the interest rate rises. In the case at hand, when I cut the interest rate in $r = 0.0942$ half, the switch point rises from 34 to 53, suggesting a negative relationship between r and N^***.** Similarly, cutting r in half again raises N^* to 85, these being bigger changes than when I cut earnings growth g in half and in quarters. Doubling, and quadrupling, the starting value of r lowers N^*, first to 21, then to 13, these being bigger changes than when I increased earnings.

7. **Conclusion**

This work has provided a specific formal sense in which the "empirical rule" of tax effects is right: deduction of tax from earnings and interest lowers the present value of future income streams for younger plaintiffs, but raises present value for older plaintiffs. The theory, based on a highly simplified tax model, is also hedged in two ways, in terms of the earnings horizon and the tax rate.

The theory can, in principle, be generalized by making a “global” mathematical comparison of with-tax and without-tax present values, rather than the “local” comparison (with tax rate near 0) provided here. Along these lines, an analysis in continuous-time would make clearer the timing of the switch-point I have discussed. I leave this agenda to future research.

References

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APPENDIX

Proof of Theorem 1

With future income growing at a constant positive rate g , and with tax on earnings and interest each at rate τ the present value of the future earning stream, computed in after-tax terms, is as follows:

(A.1)
$$PV = E_0 \sum_{t=1}^{N} \frac{(1-\tau)(1+g)^t}{(1+(1-\tau)r)^t}$$

Normalizing base earnings E_0 to equal 1, and interpreting present value as a function of the tax rate, the derivative of present value with respect to the tax rate is:

(A.2)
$$\frac{\partial}{\partial \tau} PV = \sum_{t=1}^{N} \frac{-(1+g)^t}{(1+r(1-\tau))^t} + \sum_{t=1}^{N} \frac{tr(1-\tau)(1+g)^t}{(1+r(1-\tau))^{t+1}}$$

Evaluating the derivative at $\tau = 0$, and denoting the result as PV', yields:

(A.3)
$$PV' = \sum_{t=1}^{N} \left(t\frac{r}{1+r} - 1 \right) \left(\frac{1+g}{1+r} \right)^t$$

To describe the derivative PV' as it relates to the time horizon N , define the constant:

(A.4)
$$t^{\#} = \frac{1+r}{r}$$

In the derivative formula (4.2), note that the term $t\frac{r}{1+r}-1$ is negative for $t<t^{\#}$, positive for $t>t^{\#}$, and zero for $t=t^{\#}$. Consequently, the derivative $PV^{'}$ is negative at $N=1$, falls as N increases so long as $t<t^{\#}$, and increases with y for all $t>t^{\#}$.

While the derivative $PV^{'}$ starts out negative for small N, it is not obvious whether it ever reaches 0 or a positive value when N gets large. The answer depends on whether the eventually-positive terms in the sum appearing in (4.2) are sufficient to offset the initially-negative terms. To check, consider first the case of total offset, where $g=r$. Computing:

(A.5) $$PV^{'} = \sum_{t=1}^{N}\left(t\frac{r}{1+r}-1\right)$$

(A.6) $$= N\left(\frac{N+1}{2}\frac{r}{1+r}-1\right)$$

Here $PV^{'}$ is positive for all N larger than $t^{\#}$.

If total offset fails, meaning that $g\neq r$, then with some algebra we can rewrite $PV^{'}$ as follows:

(A.7) $$PV^{'} = \frac{r}{1+r}\sum_{t=1}^{N}ta^{t}-\sum_{t=1}^{N}a^{t}$$

(A.8) $$= \frac{r}{1+r}\frac{a}{1-a}\left(\frac{1-a^{N}}{1-a}-Na^{N}\right)-\frac{a(1-a^{N})}{1-a}$$

where a is the ratio of pre-tax gross earnings growth to gross interest rate:

(A.9)
$$a = \frac{1+g}{1+r}$$

Rearranging terms in (A.8), and expressing the result in terms of g and i, we have:

(A.10)
$$PV' = \frac{g(1+g)}{(r-g)^2} + \left(\frac{1+g}{1+r}\right)^N \left(\frac{1+g}{r-g} - \left(\frac{r(1+g)}{(1+r)(r-g)}\right)\left(N + \frac{1+r}{i-g}\right)\right)$$

If $g < r$ then, for large N, (A.10) implies that PV' is eventually positive, converging to a positive value:

(A.11)
$$\lim_{y^* \uparrow \infty} PV' = \frac{g(1+g)}{(r-g)^2}$$

On the other hand, if $g > i$ then PV' diverges to ∞:

(A.12)
$$PV' \approx N\left(\frac{r(1+g)}{(1+r)(g-r)}\right)\left(\frac{1+g}{1+r}\right)^N$$

hence PV' is again positive for N sufficiently large.

Given the behavior of the derivative PV' as the earnings horizon N varies, for each income growth rate g and interest rate i there must be a threshold value y^a for N, such that the derivative PV' is negative for N less than N^* and positive for N greater than N^*. This threshold value for N is the one that makes the derivative PV' as close to zero as possible. Ideally:

(A.13) $$PV' = 0$$

In light of (A.3) and the definition of Macaulay duration D, we can rephrase the restriction (A.13) as follows:

(A.14) $$D = 1 + \frac{1}{r}$$

The threshold value N^* for the earnings horizon N is the positive integer that makes (4.13) true, or most nearly so.

To apply this characterization of the derivative PV' to the impact of small tax on the present value of earnings streams, note that present value PV is a smooth function of the tax rate, and so admits a Taylor series approximation in the neighborhood of $\tau = 0$:

(A.15) $$PV = PV^{(0)} + PV'\tau + u$$

where $PV^{(0)}$ denotes before-tax present value, PV' is the derivative of PV evaluated at $\tau = 0$, and u is a remainder term:

(A.16) $$u = \frac{\tau^2}{2} PV''(\tau^*)$$

for some τ^* in the interval $[0,\tau]$, with $PV''(\tau^*)$ the second derivative of PV with respect to τ, evaluated at $\tau = \tau^*$.

If we could bound the remainder u uniformly in N then we could apply a linear approximation to $P(\tau)$ with accuracy that is uniform in N, and this would allow us to sign the effects of (small) tax on the present value of earnings streams. However, to do so it would be necessary to bound the second derivative $PV''(\tau^*)$ uniformly in N and also uniformly in τ^* for all τ^* between 0 and τ. I will not attempt this task here, but instead note that for any given upper limit T on N we can bound the second derivative uniformly in τ^* for $N = 1, 2, ..., T$, as the second derivative is continuous for each N. With an upper bound on the earnings horizon N, linear approximation to the present value PV then yields Theorem 1.

Proof of Lemma 1

D is a weighted average of periods t, a fact that we can make more explicit as follows:

(A.17) $$D = \sum_{t=1}^{N} w_t t$$

with weights w_t:

(A.18) $$w_t = \frac{(1+h)^t}{\sum_{s=1}^{N} (1+h)^s}$$

By construction, the weights $w_1,...,w_N$ sum to 1. For larger N, weight is shifted toward later periods t, meaning that w_t rises for larger t and falls for smaller t. To check this, consider the case $N=1$ versus $N=2$. In the former case $w_1=1$. In the latter case, $w_1=(1+h)/((1+h)+(1+h)^2)$ and the weight w_2 is positive, resulting in less weight on smaller t and more weight on larger t. The comparison generally of $N=k$ versus $N=k+1$, for any counting number k, analogous.

To determine the effect of an increase in h on D, it suffices to show that the increase shifts weight w_t from smaller t to bigger t. To this end, note that the ratio of w_t in year y to w_s in any other period t is:

(A.19)
$$\frac{w_t}{w_s}=(1+h)^{t-s}$$

An increase h in therefore raises this ratio if $t>s$, as was to be shown.

Proof of Theorem 2

An increase in the earnings growth rate g raises the net growth rate h. In turn, an increase in h raises duration D by Lemma 1. If horizon N^* had exactly satisfied the switch-point equation (3) in the text before the increase in g, it can no longer do so after the increase since the left-hand side of (3) has risen but the right-hand side has not changed. To make the left-hand side equal to or less than the right-hand side, it suffices to lower D. By Lemma 1, we can do this by lowering N^*. In other words, when g goes up N^* goes down, or is at least non-decreasing. If no N^* exactly satisfies (3), and we take N^* to mean the value of N that most nearly satisfies (3), we get the same result.

Proof of Theorem 3

Differentiate (A.2) with respect to r to get:

(A.20)
$$\frac{\partial^2}{\partial\tau\partial r}PV = A+B$$

with components A and B defined as:

(A.21)
$$A=\sum_{t=1}^{N}\frac{(1+g)^t}{(1+r(1-\tau))^{t+1}}\left(t(1-\tau)\right)\left(\frac{t(1-\tau)r}{1+r(1-\tau)}-1\right)$$

(A.22)
$$B=\sum_{t=1}^{N}\frac{(1+g)^t}{(1+r(1-\tau))^{t}}\left(\frac{t(1-\tau)}{1+r(1-\tau)}-\frac{t(1-\tau)^2 r}{(1+r(1-\tau))^2}\right)$$

Rearranging terms, simplifying the result, and evaluating at $\tau=0$ yields:

(A.23)
$$\frac{\partial^2}{\partial\tau\partial r}PV=\sum_{t=1}^{N}\frac{(1+g)^t}{(1+r)^t}\left(\frac{t(t-1)r}{(1+r)^2}\right)$$

which is positive. Therefore, an (incremental) increase in r raises the value of the (first) derivative $\frac{\partial}{\partial\tau}PV$.

Consider now the switch-point time horizon N^*, and suppose that it solves equation A.2 exactly. If r goes up then so does $\frac{\partial}{\partial\tau}PV$, in which case N^* no longer solves (A.2). Also, notice that when $\frac{\partial}{\partial\tau}PV$ is evaluated at $\tau=0$, the result (A.3) is increasing in N. Therefore, those N smaller than N^* will have lower $\frac{\partial}{\partial \tau} PV$ when evaluated at $\tau = 0$. One of these N may solve (A.2) exactly, in which case this is the new (lower) value of N^*. If no N solves (A.2) exactly, then N^* need not change when r rises, but N^* remains a weakly decreasing function of r.

Table 1: Tax Effects and Earnings Horizon

horizon	pv post-tax	pv pre-tax	pv diff.	D	D – (1 + 1/i)
1	0.9	1.0	-0.1	1.0	-15.2
2	1.8	2.0	-0.2	1.5	-14.7
3	2.7	2.9	-0.2	2.0	-14.2
4	3.6	3.9	-0.3	2.5	-13.7
5	4.4	4.8	-0.4	3.0	-13.2
6	5.3	5.7	-0.4	3.5	-12.7
7	6.1	6.6	-0.5	3.9	-12.2
8	7.0	7.5	-0.5	4.4	-11.7
9	7.8	8.4	-0.6	4.9	-11.3
10	8.7	9.3	-0.6	5.4	-10.8
11	9.5	10.1	-0.6	5.9	-10.3
12	10.3	11.0	-0.7	6.3	-9.8
13	11.1	11.8	-0.7	6.8	-9.4
14	11.9	12.6	-0.7	7.3	-8.9
15	12.7	13.4	-0.7	7.7	-8.4
16	13.5	14.2	-0.7	8.2	-8.0
17	14.3	15.0	-0.7	8.7	-7.5
18	15.1	15.8	-0.7	9.1	-7.0
19	15.9	16.5	-0.7	9.6	-6.6
20	16.6	17.3	-0.7	10.0	-6.1
21	17.4	18.0	-0.7	10.5	-5.7
22	18.1	18.8	-0.6	10.9	-5.2
23	18.9	19.5	-0.6	11.4	-4.8
24	19.6	20.2	-0.6	11.8	-4.3
25	20.4	20.9	-0.5	12.3	-3.9
26	21.1	21.6	-0.5	12.7	-3.5
27	21.8	22.3	-0.5	13.1	-3.0
28	22.5	23.0	-0.4	13.6	-2.6
29	23.2	23.6	-0.4	14.0	-2.1
30	24.0	24.3	-0.3	14.4	-1.7
31	24.7	24.9	-0.3	14.9	-1.3
32	25.3	25.6	-0.2	15.3	-0.9
33	26.0	26.2	-0.1	15.7	-0.4
34	26.7	26.8	-0.1	16.1	0.0
35	27.4	27.4	0.0	16.6	0.4
36	28.1	28.0	0.1	17.0	0.8
37	28.7	28.6	0.1	17.4	1.2
38	29.4	29.2	0.2	17.8	1.7
39	30.0	29.8	0.3	18.2	2.1
40	30.7	30.3	0.4	18.6	2.5